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Louis O. Scott

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

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Louis O. Scott, Assistant Professor
Department of Finance

316 David Kinley Hall
University of Illinois
1407 W. Gregory
Urbana, IL 61801
217/333-9128

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ABSTRACT

In this paper, we examine the pricing of European call options on stocks which have variance rates that change randomly. We study continuous time diffusion processes for the stock return and the standard deviation parameter, and we find that one must use the stock and two options to form a riskless hedge. The riskless hedge does not lead to a unique option pricing function because the random standard deviation is not a traded security. One must appeal to an equilibrium asset pricing model to derive a unique option pricing function. In general, the option price depends on the risk premium associated with the random standard deviation. We find that the problem can be simplified by assuming that volatility risk can be diversified away and that changes in volatility are uncorrelated with the stock return. The resulting solution is an integral of the Black-Scholes formula and the distribution function for the variance of the stock price. We show that accurate option prices can be computed via Monte Carlo simulations and we apply the model to a set of actual prices.

OPTION PRICING WHEN THE VARIANCE CHANGES RANDOMLY: THEORY AND AN APPLICATION

The variance of stock returns plays an important role in option pricing, and it has received much attention in the empirical literature. Some researchers have developed methods for improving the accuracy of estimates of the variance from historical stock return data, while others have used option prices to recover current estimates. This work has been motivated by the observation that stock price volatility seems to change over time and that the changes are not completely predictable. The Black-Scholes model is frequently used to calculate implied standard deviations (ISD) from option prices and the ISD's are allowed to vary from one day to the next, but the underlying assumption of the model is that stock returns are lognormally distributed with a constant variance rate.[1] Other models in the literature allow the variance rate to change with some other variable such as the stock price or the underlying value of the firm. In this paper, we consider a model in which the variance rate or the standard deviation is allowed to vary randomly according to an independent diffusion process, and by constructing this model we incorporate the possibility that ISD's in option prices may change randomly from one day to the next. Before we present the model, we offer some empirical evidence which indicates that stock price volatility does change and that there is some intertemporal dependence in the volatility.

In the empirical literature on stock return distributions, there is much evidence supporting models in which the variance parameter changes randomly over time. For examples, see the papers by Blattberg and Gonedes (1974), Clark (1973), Epps and Epps (1976), and Kon (1984).

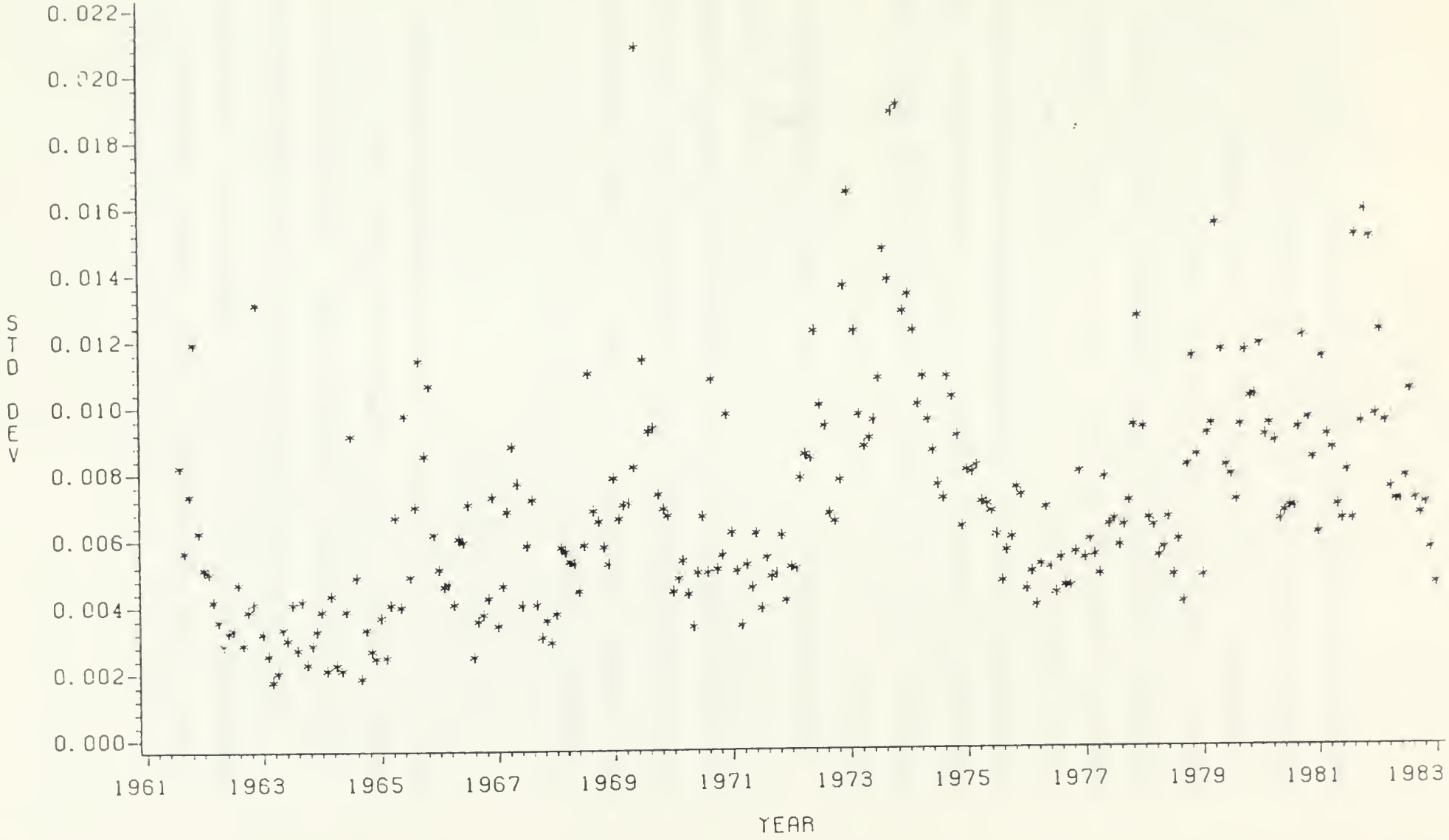
These studies and others have treated stock returns over discrete time intervals as subordinated processes: the stock return or the log of one plus the stock return is normally distributed with a directing process determining the variance each period. Blattberg and Gonedes note that if we take Brownian motion and randomize the variance of the process with an inverted gamma--2 process, the resulting distribution is a student t, which they apply to stock returns. Another approach is to use the mixture-of-normals model in which we first randomly draw mean and variance parameters from a set of possible parameter values and then generate stock returns using the normal distribution with the randomly drawn parameter values. In these applications, stock returns are independent over time: the variance parameter drawn this period is independent of the draw in any other period. In Feller's (1971, pp. 346-47) terminology, the directing process has "stationary independent increments."

If we were to compute monthly standard deviations for stock returns using daily data, we would expect the monthly estimates to be distributed randomly around the unconditional variance if the underlying stock returns are independent over time.[2] If we look at these monthly standard deviations over time, what we see is a persistent pattern. In Figure 1, we have plotted the monthly standard deviations for the value-weighted return series taken from the CRSP daily file. The sample period is July 1962 to December 1983 and the following calculation has been made for each month:

$$\hat{\sigma}_i^2 = \frac{1}{(T_i-1)} \sum_{t=1}^{T_i} (\ell n(1+R_{it}) - \hat{\mu}_i),$$

FIGURE 1

MONTHLY STANDARD DEVIATIONS, 1962–83



where $\hat{\mu}_i$ is the sample mean of $\ell n(1+R)$ for month i. In addition to the persistent pattern in Figure 1, the standard deviations have a tendency to return to an average level. We treat the 258 estimates of the monthly standard deviations as a time series and compute the first order autocorrelation coefficient. The estimate for the CRSP data is .5872. Whether we compute the non-Neumann ratio or a t-statistic using a standard error of $n^{-1/2}$, we shall reject the null hypothesis of serial independence at extremely low significance levels. Similar calculations have been made with daily returns on the S&P 500 and Digital Equipment Corporation. The autocorrelation coefficients for the monthly standard deviations are .6263 and .4529, respectively.

These observations indicate strong evidence of intertemporal dependence in stock price volatility. This phenomena cannot be explained by models in which stock returns are distributed independently over time, which is the case with the class of subordinated processes which have been frequently applied to stock returns. One possible explanation is a diffusion process of the following form:

$$dP = \alpha P_t dt + \sigma_t P_t dz,$$

where σ_t is itself a diffusion process driven by a second Wiener process. In addition, one can easily incorporate a mean-reverting tendency in the standard deviation process. The remainder of the paper is devoted to the development of an option pricing model which incorporates random variation in the volatility parameter. We focus on the valuation of European call options for non-dividend paying stocks, and

from Merton (1973), we know that the results carry over to the corresponding American call options. In Section II, we develop techniques for estimating parameters of the variance process, and in Section III, we apply the model to options on Digital Equipment Corporation (DEC) to compare the performance of the random variance model with the Black-Scholes model.

I. The Random Variance Option Pricing Model

From the observations made in the introduction, we now consider the following stochastic process for stock prices:

$$\begin{aligned} dP &= \alpha P dt + \sigma P dz_1 \\ d\sigma &= \beta(\bar{\sigma}-\sigma)dt + \gamma dz_2, \end{aligned} \tag{1}$$

where dz_1 and dz_2 are Wiener processes. Here we are assuming that the standard deviation for stock prices follows a random mean-reverting process, an Ornstein-Uhlenbeck process. If β equals zero, σ is a random walk and the unconditional variance for stock returns is infinite. The σ parameter is normally distributed and there is a possibility of negative values, but the variance will be nonnegative. At the end of this section, we derive similar results for a strictly positive process on σ. A call option on this stock will be a function of three variables: $H(P,\sigma,\tau)$, where τ is time to expiration for the option. We also make the common assumption that the riskless interest rate is constant.

We first examine this problem by forming a riskless hedge involving the stock and options to derive a partial differential equation (P.D.E.) which the option pricing function must satisfy. The introduction of a

random variance produces several complications. A dynamic portfolio with only one option and one stock is not sufficient for creating a riskless investment strategy. The problem arises because the stochastic differential for the option, dH, contains two sources of uncertainty, dz_1 and dz_2. In order to eliminate the uncertainty, we require two call options plus the stock; the two call options must have different expiration dates. This requirement does not present any difficulties because stock options trade with three expiration dates. Jones (1984) and Eisenberg (1984) have also examined option pricing models where at least two options are necessary to form a riskless hedge.

We assume the existence of the option pricing function, $H(P,\sigma,\tau)$ and use Ito's lemma to derive the stochastic differential:

$$dH = [H_1\alpha P + H_2\beta(\bar{\sigma}-\sigma) - H_3 + \frac{1}{2} H_{11}\sigma^2P^2 + H_{12}\delta\gamma\sigma P + \frac{1}{2} H_{22}\gamma^2]dt + H_1\sigma P dz_1 + H_2\gamma dz_2, \tag{2}$$

where the subscripts on H indicate partial derivatives and δ is the instantaneous correlation between dz_1 and dz_2. We form a portfolio with the stock and two calls that have different expiration dates:

$$H(\cdot,\cdot,\tau_1) + w_2H(\cdot,\cdot,\tau_2) + w_3P.$$

We set the proportions w_2 and w_3 so that the risk of the portfolio is eliminated:

$$\hat{w}_2 = - \frac{H_2(\cdot,\cdot,\tau_1)}{H_2(\cdot,\cdot,\tau_2)}$$

$$\hat{w}_3 = - H_1(\cdot,\cdot,\tau_1) + \frac{H_2(\cdot,\cdot,\tau_1)H_1(\cdot,\cdot,\tau_2)}{H_2(\cdot,\cdot,\tau_2)}.$$

After some cancellation, the return on this portfolio is

$$dH(\cdot,\cdot,\tau_1) + \hat{w}_2 dH(\cdot,\cdot,\tau_2) + \hat{w}_3 dP$$

$$= \{- H_3(\tau_1) + \tfrac{1}{2} H_{11}(\tau_1)\sigma^2P^2 + H_{12}(\tau_1)\delta\gamma\sigma P + \tfrac{1}{2} H_{22}(\tau_1)\gamma^2$$

$$- \frac{H_2(\tau_1)}{H_2(\tau_2)} [- H_3(\tau_2) + \tfrac{1}{2} H_{11}(\tau_2)\sigma^2P^2 + H_{12}(\tau_2)\delta\gamma\sigma P + \tfrac{1}{2} H_{22}(\tau_2)\gamma^2]\}dt$$

When we form the riskless hedge, we lose the expected return on the stock and the expected change in the volatility parameter. Because this portfolio has a riskless return, in equilibrium it must have a return equal to the risk-free rate. The result is the following P.D.E.:

$$- H_3(\tau_1) + \tfrac{1}{2} H_{11}(\tau_1)\sigma^2P^2 + H_{12}(\tau_1)\delta\gamma\sigma P + \tfrac{1}{2} H_{22}(\tau_1)\gamma^2 - \frac{H_2(\tau_1)}{H_2(\tau_2)} [- H_3(\tau_2)$$

$$+ \tfrac{1}{2} H_{11}(\tau_2)\sigma^2P^2 + H_{12}(\tau_2)\delta\gamma\sigma P + \tfrac{1}{2} H_{22}(\tau_2)\gamma^2] = r[H(\tau_1) + \hat{w}_2 H(\tau_2) + \hat{w}_3 P]$$

After some manipulation we have

$$[H_3(\tau_1) - \tfrac{1}{2} H_{11}(\tau_1)\sigma^2P^2 - H_{12}(\tau_1)\delta\gamma\sigma P - \tfrac{1}{2} H_{22}(\tau_1)\gamma^2 + H(\tau_1)r - H_1(\tau_1)Pr]$$

$$- \frac{H_2(\tau_1)}{H_2(\tau_2)} [H_3(\tau_2) - \tfrac{1}{2} H_{11}(\tau_2)\sigma^2P^2 - H_{12}(\tau_2)\delta\gamma\sigma P - \tfrac{1}{2} H_{22}(\tau_2)\gamma^2 \qquad (3)$$

$$+ H(\tau_2)r - H_1(\tau_2)Pr] = 0$$

The solution to the following P.D.E. is a solution to the P.D.E. in (3):

$$H_3 - \frac{1}{2} H_{11}\sigma^2 P^2 - H_{12}\delta\gamma\sigma P - \frac{1}{2} H_{22}\gamma^2 + Hr - H_1 Pr = 0$$

with the boundary condition $H(P,\sigma,0) = \max\{0,P-c\}$, where c is the exercise price of the call option. But the solution to the following P.D.E. with the same boundary conditions also solves the P.D.E. in (3):

$$H_3 - \frac{1}{2} H_{11}\sigma^2 P^2 - H_{12}\delta\gamma\sigma P - \frac{1}{2} H_{22}\gamma^2 + Hr$$

$$- H_1 Pr - H_2 b^* = 0,$$

where b* is arbitrary. Arbitrage is not sufficient for the determination of a unique option pricing function in this random variance model. An alternative view of this problem is that the duplicating portfolio for an option in this model contains the stock, the riskless bond, and another option. We cannot determine the price of a call option without knowing the price of another call on the same stock, but that is precisely the function that we are trying to determine.

To derive a unique option pricing function, we must rely on an equilibrium asset pricing model. This is the approach used by Hull and White (1986), and we apply their technique. From the stochastic differential for the option price, we know that dH depends on two random variables, dP and dσ. By applying either an intertemporal asset pricing model or a continuous-time version of Ross's (1976, 1977) arbitrage pricing theory, we have the following equation for the expected return on the option:

$$E\left(\frac{dH}{H}\right) = r + \frac{H_1 P}{H}(\alpha - r) + \frac{H_2}{H}\lambda^*,$$

where $(\alpha - r)$ is the risk premium on the stock and λ^* is the risk premium associated with $d\sigma$. The expected return on the option is also determined by the dt term in (2). Equating these two expressions for the expected return, we have

$$-H_3 + \frac{1}{2}\sigma^2 P^2 H_{11} + \delta\gamma\sigma P H_{12} + \frac{1}{2}\gamma^2 H_{22} - rH + rPH + H_2[\beta(\bar{\sigma}-\sigma) - \lambda^*] = 0. \tag{4}$$

The P.D.E. in (4) with the boundary condition has a unique solution and it is easy to show that this solution also satisfies the P.D.E. in (3). The expected return on the stock does not influence the value of the option, but in general, the expected change and the risk premium associated with the volatility parameter do.

By applying the results in Lemma 4 of Cox, Ingersoll, and Ross (1985), we have the following solution for the option pricing function:

$$H(P,\sigma,t;r,c) = \hat{E}(e^{-rt}\max\{0,P_t-c\}\,|\,P_0,\sigma_0), \tag{5}$$

where $\hat{E}$ is a risk-adjusted expectation. For the risk-adjustment, we reduce the mean parameters of dP and $d\sigma$ by the corresponding risk premia. For the stock return, we replace α with the risk-free rate, r. For the standard deviation, we use $[\beta(\bar{\sigma}-\sigma) - \lambda^*]$ in place of $\beta(\bar{\sigma}-\sigma)$. By following Karlin and Taylor (1981, pp. 222-24), we can derive the backward equation for the function in (5) and show that it solves the P.D.E. in (4) with these adjustments on the dP and $d\sigma$ processes. This result is demonstrated in the appendix.

The option pricing function in (5) is a general solution to this random variance problem. To make this model operational, we need the parameters of the σ process, the risk premium λ^*, and the instantaneous correlation coefficient between the stock return and $d\sigma$. Given these parameters and the current value of σ, one can use the Monte Carlo simulation method described in Boyle (1977) to compute option prices. The model can be simplified if λ^* and δ are zero.[3] The risk premium is zero if the volatility risk of the stock is diversifiable (or if $d\sigma$ is uncorrelated with the marginal utility of wealth). If the risk premium is zero, then the change in σ should be uncorrelated with the stock return. By contrast, if the risk premium is non-zero, then the change in σ should be correlated with the stock return. We make the following argument. Assume that there is a market volatility factor and that the stock's volatility is positively related to the market volatility factor. If this is true, volatility risk cannot be completely diversified away. For the market portfolio, there is a common belief that there is a positive relationship between the risk premium and volatility. If there is an unexpected increase in market volatility, the risk premium on the market portfolio rises, but at the same time the value of the market portfolio would normally decline. We would get a drop in the values of most stocks and this suggests a negative correlation between stock price changes and its volatility. Our a priori reasoning suggests a negative covariance between $d\sigma$ and dP and a negative covariance between $d\sigma$ and changes in the value of the market portfolio. This latter relationship implies a negative risk premium, λ^*, for σ. The significance of the parameters λ^* and δ is an empirical issue that we

do not explore in this paper. These parameters may or may not be significant, but by setting them equal to zero we can simplify the model and significantly reduce the costs of the Monte Carlo simulations. With $d\sigma$ uncorrelated with dP, we can use the conditional distribution of the stock return given the variance process.

We develop the distribution of the stock price at expiration with $\lambda^* = 0$ and $\delta = 0$. Applying the results on stochastic calculus in Karlin and Taylor (pp. 368-75), we have the following solution to the stochastic differential for stock prices:

$$P_t = P_0 \exp\{\int_0^t (r - \frac{1}{2}\sigma^2_{(s)})ds + \int_0^t \sigma_{(s)} dz_1(s)\}.$$

Next we examine the distribution of P_t conditional on both P_0 and the path of σ, $\{\sigma_s\}$ for $0 \leq s \leq t$. This conditional distribution is lognormal and the expectation is

$$E(P_t | P_0, \{\sigma_s\}) = P_0 e^{rt}$$

Then taking the expectation of $E(P_t | P_0, \{\sigma_s\})$ over the distribution of $\{\sigma_s\}$ for $0 \leq s \leq t$, we get the same result and the expected return on the stock equals the riskless return. We find the following integral to be a useful parameter: $V = \int_0^t \sigma^2_{(s)} ds$, which is of course random. Our distribution for stock prices conditional on $\{\sigma_s\}$ is lognormal:

$$\ln(P_t/P_0) \sim N(rt - \frac{1}{2}V, V).$$

Now apply the results of Smith (1976, pp. 15-16),

$$e^{-rt} E(\max\{0, P_t - c\} | P_0, V) = P_0 N(d_1) - c e^{-rt} N(d_2),$$

where $d_1 = \dfrac{\ln(P_0/c) + rt + \frac{1}{2}V}{\sqrt{V}}$

and $d_2 = d_1 - \sqrt{V}$.

This result is essentially the Black-Scholes formula with V in place of $\sigma^2 t$. To finish the problem, we need to integrate this formula over the distribution of V. From equation (1) and the expression above, V depends on σ_0, t, β, $\bar{\sigma}$, and γ. The resulting form of the option pricing function is

$$H(P_0,\sigma_0,t;r,c,\beta,\bar{\sigma},\gamma) = \int_0^\infty [P_0 N(d_1) - ce^{-rt}N(d_2)]dF(V;t,\sigma_0,\beta,\bar{\sigma},\gamma). \quad (6)$$

This integral converges because F is a distribution function, and the function inside the brackets is bounded given the values of P_0, c, r, and t. The functions $N(d_1)$ and $N(d_2)$ are bounded by zero and one. If we could analytically determine the density function for V, calculation of option prices for this model would involve numerical integration of the Black-Scholes formula, and we would call such a solution a quasi closed-form.[4] The distribution of V for the σ process in (1) is quite complicated because the integral is the sum of the squares of correlated normal variates. The option pricing function involves the expectation of a function of V, g(V), and one might be able to develop some accurate approximations by using a function of the mean and variance of V, which can be analytically determined.[5]

Our approach is to compute option prices by Monte Carlo simulations. Let $g(V) = P_0 N(d_1) - ce^{-rt}N(d_2)$, and our solution is E(g(V)) taken over the distribution of V. Given that this moment exists, as we have argued,

one can simulate values of V and g(V) and compute the sample mean for simulated values of g(V). As the sample size gets large, we know that the sample mean is closing in on E(g(V)), our option price, because the sample mean converges in probability to the expected value. An empirical question remains regarding the sample size necessary for computing accurate option prices from the model. One advantage of our approach is that we do not need to simulate both V and P_t; we need to simulate only V and this substantially reduces the sample size or number of trials required for a given level of accuracy.

We have also developed the model for a lognormal process on the σ parameter, namely that lnσ is an Ornstein-Uhlenbeck process. The stochastic differential for σ is

$$d\sigma = \sigma[\frac{1}{2}\gamma^2 - \beta(\ln\sigma - \bar{a})]dt + \gamma\sigma dz_2, \tag{7}$$

where $\bar{a}$ is the mean reverting value for lnσ.[6] We apply the same approach used for the first model: we use an equilibrium asset pricing model and set δ and λ* equal to zero. The resulting P.D.E. is

$$-H_3 + \frac{1}{2}H_{11}\sigma^2P^2 + \frac{1}{2}H_{22}\gamma^2\sigma^2 - Hr + H_1Pr + H_2\sigma[\frac{1}{2}\gamma^2 - \beta(\ell n\sigma - \bar{a})] = 0 \tag{8}$$

with the same boundary condition. The solution to this P.D.E. is identical to the solution in (5), except the distribution for V is different. The integral V now involves the summation of correlated lognormal variates, and the simulation of V must be modified appropriately.

II. Estimating the Parameters of the Variance Process

In order to compute option prices from the models in the previous section, we need values for σ_0 and the parameters of the σ process.

We first consider the estimation of the parameters of the σ process from data on the stock returns. Because the volatility parameter, σ_0, changes randomly, its estimation will be more difficult. A common approach in the empirical literature on option pricing is to use actual option prices to infer the values of σ_0. This approach is used in the next section where we apply the model to a series of actual call option prices. At the end of this section, we outline briefly two Kalman filter models that might be used to estimate current values of σ.

For the volatility process in equation (1), the fixed parameters are β, $\bar{\sigma}$, and γ. One approach to estimating these parameters would be to determine the unconditional distribution of stock returns as a function of α, β, $\bar{\sigma}$, and γ, and then apply the method of maximum likelihood. The problem with the maximum likelihood estimation is that stock returns are dependent over time in this model and the joint distribution for a sample of observations would be very difficult to derive. Our approach is to use the method of moments to jointly estimate the parameters of the stock return process.

Because the data on stock prices are generally available at fixed points in time, we apply a discrete time approximation to the volatility process. Over short time intervals, the distribution of stock returns conditional on the volatility parameter is lognormal and we have a process

$$\Delta \ln P_t = \alpha \Delta t + \sigma_t \Delta z,$$

where Δz is $N(0,\Delta t)$. From the Ornstein-Uhlenbeck process for σ, we can derive the following equation for σ at discrete points equally spaced:

$$\sigma_t = e^{-\beta}\sigma_{t-1} + \bar{\sigma}(1-e^{-\beta}) + \varepsilon_t,$$

where ε_t is normal with mean zero and variance

$$\text{Var}(\varepsilon_t) = \frac{\gamma^2(1-e^{-2\beta})}{2\beta}$$

The variance of stock returns, $\Delta \ln P_t$, over an interval is $\int_0^t \sigma^2_{(s)} ds$. For small intervals Δt, we use $\Delta t \sigma^2_{(s)}$, where s is the midpoint of the interval. This approximation can be made as accurate as desired by decreasing the size of the interval. Because stock returns are available on a daily basis, we use a day as our time interval and assume that during the day the variation in σ_t is small enough so that we may use a discrete time first order autoregressive process for σ that corresponds to the σ process above at fixed points:

$$\sigma_t = a + \rho\sigma_{t-1} + \varepsilon_t.$$

For stock returns we have $\Delta \ln P_t = \alpha + \sigma_t u_t$, where u_t is standard normal and σ_t is the standard deviation per day. The parameters α, a, ρ, and σ^2_ε can now be estimated from various moments of $\Delta \ln P_t$. For α, we use the sample mean and then define the series $x_t \equiv \Delta \ln P_t - \hat{\alpha}$. We then use estimates of the variance, the fourth moment, and the first order autocovariance of x^2_t and x^2_{t-1} to recover estimates of the remaining three parameters. Given the AR process for σ, we have $\sigma_t \sim N(\frac{a}{1-\rho}, \frac{\sigma^2_\varepsilon}{1-\rho^2})$.

$$E(x^2_t) = (\frac{a}{1-\rho})^2 + \frac{\sigma^2_\varepsilon}{1-\rho^2}$$

The sample variance is used to estimate $E(x^2_t)$.

$$E(x_t^4) = 9\frac{\sigma_\varepsilon^4}{(1-\rho^2)^2} + 18(\frac{a}{1-\rho})^2\frac{\sigma_\varepsilon^2}{1-\rho^2} + 3(\frac{a}{1-\rho})^4$$

We use the sample fourth moment of x_t to estimate $E(x_t^4)$. The following observation is useful:

$$9(E(x_t^2))^2 - E(x_t^4) = 6(\frac{a}{1-\rho})^4$$

Finally,

$$\text{Cov}(x_t^2, x_{t-1}^2) = 2\rho^2(\frac{\sigma_\varepsilon^2}{1-\rho^2})[\frac{\sigma_\varepsilon^2}{1-\rho^2} + 2(\frac{a}{1-\rho})^2]$$

$$= 2\rho^2[(E(x_t^2))^2 - (\frac{a}{1-\rho})^4]$$

$$= \rho^2[\frac{E(x_t^4)}{3} - (E(x_t^2))^2].$$

By plugging in the sample estimates, we get

$$\hat{\rho} = \sqrt{\frac{\text{Cov}(x_t^2,\ x_{t-1}^2)}{[\frac{E(x_t^4)}{3} - (E(x_t^2))^2]}}$$

$$\hat{a} = (1-\rho)\sqrt[4]{\frac{9(E(x_t^2))^2 - E(x_t^4)}{6}}$$

$$\hat{\sigma}_\varepsilon^2 = (1-\rho^2)[E(x_t^2) - \frac{a^2}{(1-\rho)^2}]$$

It is possible to use these parameter estimates to compute estimates for β, $\bar{\sigma}$, and γ, but we use a, ρ, and σ_ε in the discrete-time simulation of the σ process. It is worth noting that these parameter estimates depend on the excess kurtosis of stock returns. If $E(x_t^4) \leq 3[E(x_t^2)]^2$,

then the parameter estimation breaks down. Since these estimators are functions of sample moments, one could set this estimation up as Hansen's (1982) general method of moments estimator and work out expressions for standard errors of the estimates, but we leave this exercise to future research.

For the lognormal process in equation (7), we use the following first order AR process:

$$\ln\sigma_t = a + \rho \ln\sigma_t + \varepsilon_t,$$

where $\varepsilon_t \sim N(0,\sigma_\varepsilon^2)$. This process is a discrete approximation for the Ornstein-Uhlenbeck process on $\ell n\sigma_t$. After computing the sample mean for $\Delta \ln P_t$, we again work with $x_t \equiv \Delta \ln P_t - \hat{\alpha} = \sigma_t u_t$, where u_t is standard normal. With this process, the second and fourth moments of x_t are:

$$E(x_t^2) = \exp\{2(\frac{a}{1-\rho}) + 2(\frac{\sigma_\varepsilon^2}{1-\rho^2})\}$$

$$E(x_t^4) = 3 \exp\{4(\frac{a}{1-\rho}) + 8(\frac{\sigma_\varepsilon^2}{1-\rho^2})\}.$$

From the sample moments, we have estimators for $(\frac{\sigma_\varepsilon^2}{1-\rho^2})$ and $(\frac{a}{1-\rho})$. There are several methods for estimating ρ. A simple approach is to observe that $\ell n|x_t| = \ell n\sigma_t + \ell n|u_t|$ and

$$Cov(\ell n|x_t|, \ell n|x_{t-1}|) = \rho(\frac{\sigma_\varepsilon^2}{1-\rho^2}).$$

With estimates of this covariance and $(\frac{\sigma_\varepsilon^2}{1-\rho^2})$, we can identify an estimate for ρ. Then given $\hat{\rho}$, we can compute estimates of a and σ_t^2 from the second and fourth moments.

Estimating the σ parameter for either of these processes is considerably more difficult. One possibility is to use the Kalman filter model for estimating the value of an unobservable variable. For the first case where σ is an Ornstein-Uhlenbeck process, we shall require additional information. Several studies have presented evidence that stock return volatility is correlated with volume, measured as either shares traded or number of transactions.[7] One view of this relationship is that there is an underlying parameter related to the rate at which information hits the market, which determines both volatility and volume. The following is one plausible model:

$$\ln v_t = b + d\sigma_t + \eta_t,$$

where v_t is volume and η_t is either white noise or a moving average process, independent of σ_t. We add to this model our first order AR process for σ_t. Identification of an ARMA process for volume is not sufficient for estimating all of the parameters in this equation, but we can identify these parameters if we use the covariance of $\ln v_t$ with the square of the stock return. After estimating the necessary parameters, one can use the Kalman filter algorithm to compute estimates of σ_t from the volume data.

For the lognormal process, we observe that we have a linear model in $\ln|x_t| = \ln\sigma_t + \ln|u_t|$. First we need $E(\ln|u_t|)$, which is $-.635181421...$, and we have a Kalman filter model:

$$\ln|x_t| = b^* + \ln\sigma_t + e_t,$$

where $b^* = E(\ln|u_t|)$. $Var(e_t) = Var(\ln|u_t|) = \pi^2/8$. Even though e_t is not normally distributed, the Kalman filter estimator for $\ln\sigma_t$ is a minimum mean squared error estimator within the class of linear estimators. Here we use the Kalman filter algorithm to compute estimates of $\ln\sigma_t$ from stock return data, specifically from $\ln|\Delta\ln P_t - \hat{\mu}|$.

One other approach is to assume that the market uses a wide range of data and information and is able to determine the values of the variance process. If this were true, then option prices would reflect the unobservable σ process, and researchers could then compute ISD's which force the model prices to equal actual option prices. This practice is widely employed in the options literature and we use it in the next section to examine the ability of this model to fit actual option prices.

III. An Application of the Random Variance Option Pricing Model

In this section we use both the random variance model of Section I and the Black-Scholes model to compute prices for call options on Digital Equipment Corporation (DEC) for the period July 1982 - June 1983. We have chosen DEC because it does not pay cash dividends and it allows us to circumvent the dividend problem in this study. DEC is also a volatile stock. Option prices and stock prices for DEC have been collected at weekly intervals from the Wall Street Journal, so that we have 52 days of option prices. We use closing prices every Thursday except for Thanksgiving when we use Friday prices. Treasury bill prices are used to impute interest rates. For each option, we choose a T-bill that matures close to the option's expiration date and compute the corresponding continuously compounded yield.

We use daily stock returns for the period 1974 to June 1982 and the method of moments estimator in Section II to compute estimates of a, ρ, and σ_ε. The sample size is 2150. For this application, we have used the Ornstein-Uhlenbeck process for σ and the discrete approximation: $\sigma_t = a + \rho\sigma_{t-1} + \varepsilon_t$. The three sample moments estimated from the stock returns are: sample variance, $E(x_t^2) = .4050793 \times 10^{-3}$, $E(x_t^4) = .8221057 \times 10^{-6}$, and $Cov(x_t^2, x_{t-1}^2) = .6817389 \times 10^{-7}$. The sample kurtosis, $E(x_t^4)/(E(x_t^2))^2$, equals 5.01. The corresponding parameter estimates for the discrete σ process are $\rho = .7874$, $a = .003863$, and $\sigma_\varepsilon = .005329$.

To estimate the σ parameter for different days, we have used a technique common in the literature. We use at the money options and find the value of σ which provides the best fit of the model to actual option prices. Formally we minimize the sum of squared errors between the model and actual prices:

$$\min_{\sigma_t} \ell = \sum_{i=1}^{N} (w_{it} - H_{it}(\sigma_t))^2$$

where w_{it} is the actual price for option i on day t and $H_{it}(\sigma_t)$ is the corresponding model price as a function of σ_t. The nonlinear minimization technique that we employ uses first derivatives and the expected value of the second derivative.[8] Given a starting value, the iteration proceeds as follows:

$$\sigma_{t,i} = \sigma_{t,i-1} - \frac{\frac{\partial \ell}{\partial \sigma}}{D}$$

where $D = 2 \sum_{i=1}^{N} \left(\frac{\partial H_{it}(\sigma_t)}{\partial \sigma}\right)^2$. Note that $\frac{\partial^2 \ell}{\partial \sigma} = D + \sum_{i=1}^{N} (H_{it}(\sigma_t) - w_{it}) \frac{\partial^2 H_{it}(\sigma_t)}{\partial \sigma^2}$.

We find that this technique converges quite rapidly for our problem: typically three to four iterations for the random variance model and

one or two iterations for the Black-Scholes model. For at the money options, we use those which have exercise prices within $5 of the stock price.

Given the σ_t estimates, we compute model prices for the remaining in-the-money and out-of-the-money options and compare the model prices to actual prices. The same procedure is repeated with the Black-Scholes formula: first we estimate the daily σ_t values by minimizing the sum of squared errors between actual prices and Black-Scholes prices, and then we use the σ estimates to compute Black-Scholes prices for the remaining options. It should be noted that there is an internal inconsistency in this application of the Black-Scholes model. The Black-Scholes model is derived under the assumption that the variance rate is constant or at most a deterministic function of time. We then use the model to calculate ISD's, but allow these to vary from one day to the next. We make an additional set of calculations for the Black-Scholes model with a constant variance rate; we use the average of the daily ISD's computed from the Black-Scholes model.

For the random variance model, we have found that the ISD's are very sensitive to the value of ρ used in the simulations. For a low value of ρ such as the estimate of .7874, we get extreme variation in the ISD's. Some initial checks on the method of moments estimation indicate that the estimates of $(\frac{a}{1-\rho})$ and $(\frac{\sigma_\varepsilon^2}{1-\rho^2})$ are reliable, but we do not get precise estimates for ρ. For this reason, we fixed $(\frac{a}{1-\rho})$ and $(\frac{\sigma_\varepsilon}{\sqrt{1-\rho^2}})$ at their estimated values of .018175 and .008645914, respectively, and varied ρ in the simulations. To control the computing expense, we

have used the first 26 days of prices on at-the-money options and a simple grid search to determine the ρ value which yields the best fit with the random variance model. We examined values of .95, .98, .99, and .999, and found that $\rho = .99$ provides the best fit. In all the subsequent calculations, we use $\rho = .99$, $a = .018175\ (1-\rho)$, and $\sigma_\varepsilon = .008645914\ \sqrt{1-\rho^2}$. The higher ρ values implicit in option prices could also be the result of a negative risk premium, λ^*.

Various calculations with these models are presented in Tables I-III. For the Monte Carlo simulations of the random variance model, we use the antithetic variate method and 1000 trials to compute each option price.[9] To check the accuracy of the simulation method, we have computed prices and large sample standard errors for deep out-of-the-money, at-the-money, and deep in-the-money options. The results are contained in Table I. With 1000 trials, we are able to reduce the standard error of the estimate to \$.0075 in the worst case, which is a deep in-the-money option with 270 days to expiration (approximately 9 months). This corresponds to a 95% confidence interval of $\pm$ 1 1/2 cents.

In Table II, we present the implied standard deviations computed from both the random variance model and the Black-Scholes model for a 52 week period from July 1, 1982, to June 23, 1983. Both models are computed with trading days to expiration so that the ISD's are consistent with standard deviations computed from stock returns. In the last column, we show the monthly standard deviations; these numbers reflect the square root of an estimate of the average of the daily variance rates during the month and contain sampling error. With only twelve months of data in the table, one cannot make any conclusions as

Table I

Option Prices from the Random Variance Model

1000 trials per estimate
Exercise price = $50
r = .09 per annum

$\sigma_0 = .025$
$\rho = .99$
$a = .018175(1-\rho)$
$\sigma_\varepsilon = .008646\sqrt{1-\rho^2}$

Stock Price	Days to Expiration	Option Prices by Monte Carlo Simulation of Equation	Standard Error of Estimate
$25	30	3.88×10^{-6}	3.67×10^{-7}
	60	.001	.0001
	90	.009	.0003
	120	.027	.0008
	150	.056	.0014
	180	.094	.0019
	210	.141	.0025
	240	.195	.0029
	270	.256	.0034
$50	30	2.819	.0003
	60	3.989	.0011
	90	4.883	.0022
	120	5.637	.0031
	150	6.304	.0039
	180	6.912	.0044
	210	7.479	.0049
	240	8.013	.0054
	270	8.518	.0057
$75	30	25.373	.0001
	60	25.800	.0011
	90	26.282	.0026
	120	26.785	.0040
	150	27.291	.0051
	180	27.790	.0059
	210	28.282	.0066
	240	28.767	.0071
	270	29.240	.0075

Table II

Implied Standard Deviations (ISD's)
Estimated from Prices on Options at-the-money
Digital Equipment Corporation

Date	Number of Options Used for Estimates	Random Variance Model	Black-Scholes Model	Estimated Monthly Standard Deviations
7/1/82	4	.01915	.01938	
8	5	.01718	.01790	
15	4	.01956	.01960	.01670
22	4	.02126	.02072	
29	6	.01812	.01861	
8/15/82	6	.02176	.02061	
12	6	.02413	.02215	.02740
19	6	.02393	.02197	
26	5	.02359	.02146	
9/2/82	6	.02269	.02112	
9	3	.03032	.02566	
16	5	.02332	.02172	.02452
23	6	.02428	.02208	
30	5	.02685	.02411	
10/7/82	6	.02135	.02025	
14	4	.02834	.02484	.03298
21	6	.02618	.02297	
28	3	.03261	.02709	
11/4/82	6	.02607	.02312	
11	3	.02826	.02430	.02984
18	2	.02948	.02600	
26	4	.03256	.02757	
12/2/82	3	.03209	.02706	
9	3	.03391	.02813	
16	5	.03346	.02852	.02661
23	5	.03172	.02711	
30	5	.03076	.02697	
1/6/83	6	.02258	.02096	
13	3	.02953	.02587	.02999
20	2	.02518	.02300	
27	3	.02968	.02497	

Table II (continued)

2/3/83	3	.03077	.02585	
10	3	.02876	.02477	.02223
17	3	.02643	.02350	
24	3	.02996	.02562	
3/3/83	3	.02614	.02335	
10	3	.02582	.02333	
17	3	.02333	.02178	.01725
24	3	.02112	.02043	
31	3	.02155	.02085	
4/7/83	3	.01988	.01968	
14	2	.01772	.01821	.02038
21	2	.02212	.02125	
28	3	.02490	.02253	
5/5/83	3	.02181	.02077	
12	3	.02208	.02089	.02213
19	3	.02549	.02320	
26	2	.02356	.02199	
6/2/83	3	.02631	.02353	
9	3	.02543	.02290	.02583
16	3	.02767	.02435	
23	3	.02486	.02283	

to which model provides a better estimate of the underlying variance rate.

In Table III, we present summary statistics for the three different models. Using 728 options that are either in-the-money or out-of-the-money, we compute the sum of squared errors and mean squared errors for each model. The random variance model outperforms the Black-Scholes models with daily variance rates that change: The mean squared error for the random variance model is 8.7% less than that for the Black-Scholes model. Even though there is a difference in the mean squared error, we have not attempted a formal test. Such a test would be difficult to construct because the errors in fitting the option prices are likely to be correlated both across options and over time. The Black-Scholes model with a single variance estimate performs quite poorly in comparison with the other two models, and we can conclude that this model is clearly rejected by the data.

Some researchers have observed that there is a strong bias in the Black-Scholes model with respect to out-of-the-money options. In Figures 2 and 3, we have plotted percentage errors against a measure of whether the option is in or out-of-the-money. The percentage error is

$$\frac{w_{it} - H_{it}(\hat{\sigma}_t)}{H_{it}(\sigma_t)},$$

where $H_{it}(\hat{\sigma}_t)$ is the model price using the estimated ISD, and

$$m_{it} = \frac{S_t - X_i e^{-r\tau}}{X_i e^{-r\tau}}$$

is the measure of whether the option is in or out-of-the-money. This measure has been used by MacBeth and Merville (1979). Figure 2 is the

Table III

Digital Equipment Corporation
July 1982 to June 1983
52 Trading Days, 728 Option Prices

$$e_{it} = w_{it} - H_{it}(\hat{\sigma}_t)$$

	Sum of Squared Errors	Mean Squared Error
Random Variance Model	539.4189	.7410
Black-Scholes Model	591.1685	.8120
Black-Scholes Model with Single Variance Estimate	971.4013	1.3343

NOTE: Option prices were collected for Thursday of each week.

FIGURE 2

RANDOM VARIANCE MODEL

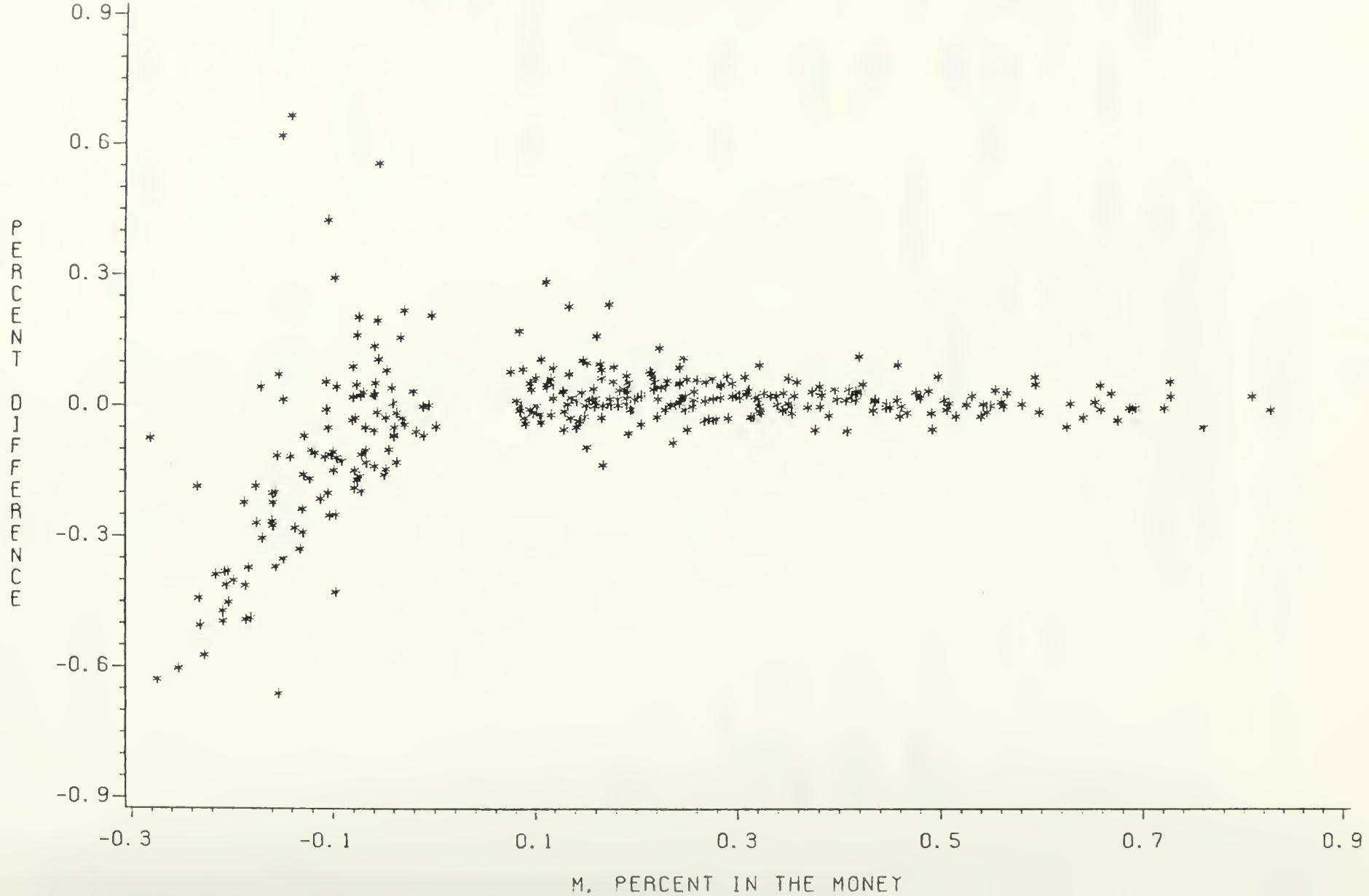
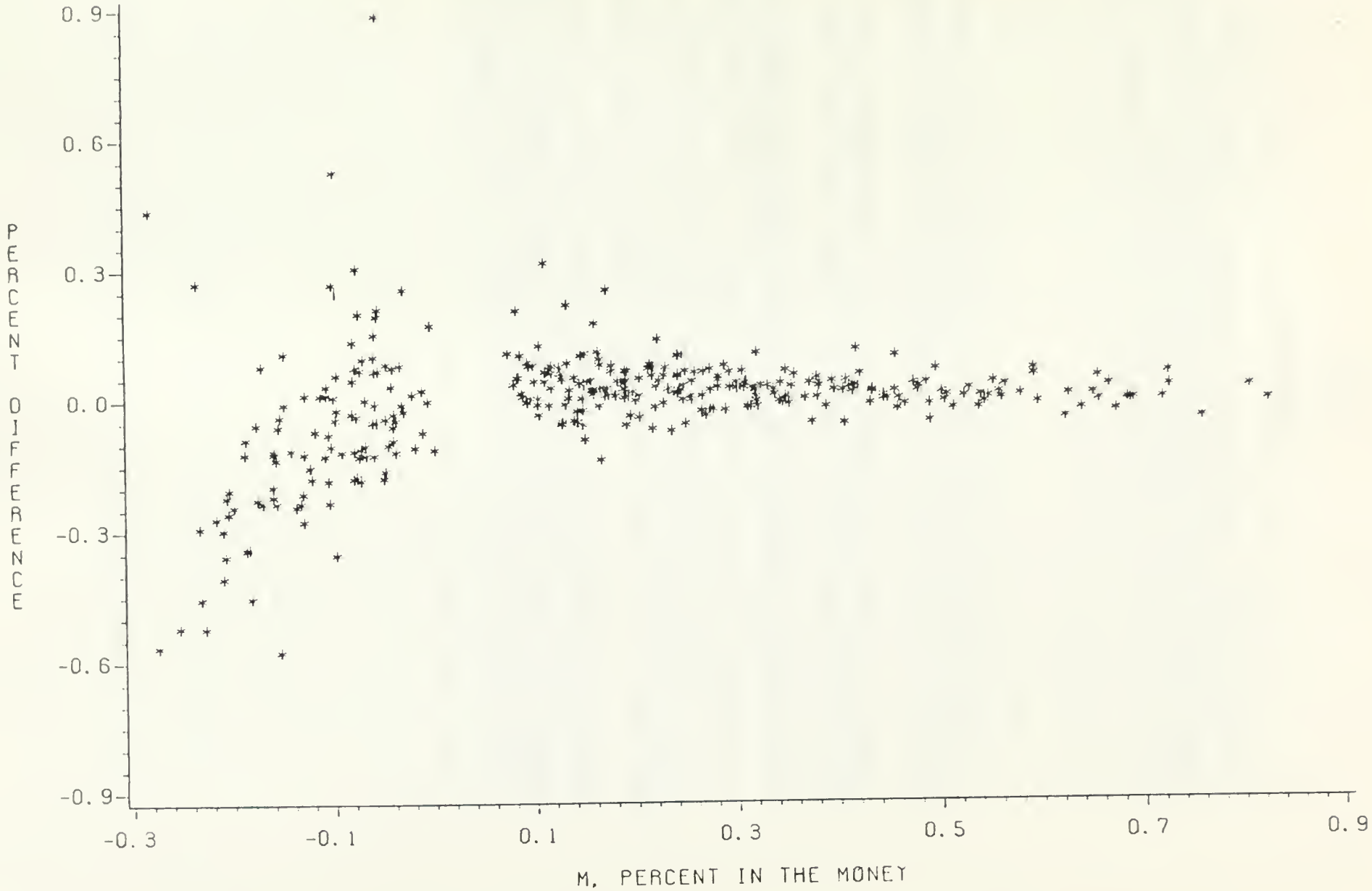


FIGURE 5

BLACK–SCHOLES MODEL



plot for the random variance model and Figure 3 is the plot for the Black-Scholes model. The graphs are very similar to those in MacBeth and Merville, and it is apparent that this bias also exists in the random variance model. Both models tend to overprice out-of-the-money options.

IV. Conclusions

We have developed an option pricing model which allows the variance parameter to change randomly, and although we are not able to develop an analytical formula, we do derive a model which can produce accurate estimates of option prices via the method of Monte Carlo simulations. We have presented evidence in the introduction that stock returns are not independent over time and that the variance of stock returns changes randomly, possibly with a mean reverting tendency. The option pricing model that we develop uses a continuous time diffusion process that captures this observed behavior for stock return volatility. We have examined two possible specifications of the variance process, and using a limited sample we find that the random variance model is marginally better at explaining actual option prices.

FOOTNOTES

[1]It is relatively easy to allow the variance parameter to vary as a deterministic function of time and derive an option pricing formula similar to the Black-Scholes model. Geske and Roll (1984) have recently noted that a nonstationary variance may account for some of the biases observed in empirical applications of the Black-Scholes model.

[2]And of course, we require the existence of the unconditional variance.

[3]Hull and White also use this simplifying assumption.

[4]Note that the Black-Scholes formula also involves numerical integration to compute $N(d_1)$ and $N(d_2)$. Here we would have one added dimension to the numerical integration.

[5]Levy and Markowitz (1979), for example, have found that functions of means and variances provide good approximations for expected utility.

[6]We derive this stochastic differential by letting another variable x be an Ornstein-Uhlenbeck process: $dx = \beta(\bar{a}-x)dt + \gamma dz_2$. Let $\sigma_t = \exp\{x_t\}$ and apply Ito's lemma to get $d\sigma$.

[7]See papers by Harris (1985) and Tauchen and Pitts (1983).

[8]If the objective function were a likelihood function, the technique would be called the method of scoring.

[9]For a discussion of the antithetic variate method and other Monte Carlo techniques, see Boyle (1977). The option prices have been computed in CDC Fortran 5. We use the most efficient CDC Fortran compiler and we use the polar coordinate method for generating standard normal random variates.

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APPENDIX

In this appendix, we show that the option pricing function in (5) solves the P.D.E. in (4) subject to the boundary condition. To do this, we derive the backward equation for a Kac functional with two state variables and show that with appropriate modifications on the stock return and σ processes, we have the P.D.E. in (4). Our derivation follows the one in Karlin and Taylor (1981, pp. 222-24) for a Kac functional with one state variable.

Let $\quad w(P_0,\sigma_0,t) = E_{P_0,\sigma_0}[\exp\{-rt\}g(P_{(t)})]$

where $g(P_{(t)})$ is bounded. We use the stochastic differentials for P and σ in equation (1).

$$e^{-rt}g(P_{(t)}) = e^{-rh}e^{-r(t-h)}g(P_{(t)})$$

$$= (e^{-rh}-1)e^{-r(t-h)}g(P_{(t)}) + e^{-r(t-h)}g(P_{(t)})$$

Applying Taylor's Theorem, we get $e^{-rh}-1 = -hr + o(h)$.

$$w(P_0,\sigma_0,t) = E_{P_0,\sigma_0}\{E_{P(h),\sigma(h)}[(1-hr+o(h))e^{-r(t-h)}g(P_{(t)})]\}$$

$$= E_{P_0,\sigma_0}\{(1-hr+0(h))E_{P(h),\sigma(h)}[e^{-r(t-h)}g(P_{(t)})]\}$$

Now noting that $w(P(h),\sigma(h),t-h) = E_{P(h),\sigma(h)}\{e^{-r(t-h)}g(P_{(t)})\}$, we have

$$w(P_0,\sigma_0,t) = E_{P_0,\sigma_0}[(1-hr)w(P(h),\sigma(h),t-h)] + o(h) \qquad (A\text{-}1)$$

We now apply a Taylor series expansion to $w(\cdot,\cdot,t-h)$ about the state variables evaluated at $P(h) = P_0$ and $\sigma(h) = \sigma_0$,

$$w(\cdot,\cdot,t-h) = w(P_0,\sigma_0,t-h) + (P(h)-P_0)\frac{\partial w(P_0,\sigma_0,t-h)}{\partial P}$$

$$+ (\sigma(h)-\sigma_0)\frac{\partial w(P_0,\sigma_0,t-h)}{\partial \sigma} + \frac{1}{2}(P(h)-P_0)^2\frac{\partial^2 w(P_0,\sigma_0,t-h)}{\partial P^2}$$

$$+ \frac{1}{2}(\sigma(h)-\sigma_0)^2\frac{\partial^2 w(P_0,\sigma_0,t-h)}{\partial \sigma^2}$$

$$+ (P(h)-P_0)(\sigma(h)-\sigma_0)\frac{\partial^2 w(P_0,\sigma_0,t-h)}{\partial P \partial \sigma} + o(h)$$

Now plug this into (A-1) and take expectations:

$$w(P_0,\sigma_0,t) = (1-hr)[w(P_0,\sigma_0,t-h) + \alpha P_0 h\frac{\partial w}{\partial P}$$

$$+ \beta(\bar{\sigma}-\sigma_0)h\frac{\partial w}{\partial \sigma} + \frac{1}{2}\sigma_0^2 P_0^2 h\frac{\partial^2 w}{\partial P^2} + \delta\gamma\sigma_0 P_0\frac{\partial^2 w}{\partial P\partial\sigma} + \frac{1}{2}\gamma^2 h\frac{\partial^2 w}{\partial\sigma^2}]$$

$$+ o(h)$$

$$w(P_0,\sigma_0,t) - w(P_0,\sigma_0,t-h) = -hrw(P_0,\sigma_0,t-h) + \alpha P_0 h\frac{\partial w}{\partial x}$$

$$+ \beta(\bar{\sigma}-\sigma_0)h\frac{\partial w}{\partial \sigma} + \frac{1}{2}\sigma_0^2 P_0^2 h\frac{\partial^2 w}{\partial P^2} + \delta\gamma\sigma_0 P_0 h\frac{\partial^2 w}{\partial P\partial\sigma}$$

$$+ \frac{1}{2}\gamma^2 h\frac{\partial^2 w}{\partial\sigma^2} + o(h)$$

Now divide by h and let h go to zero, noting that $\lim_{h\to 0}\frac{o(h)}{h} = 0$. This yields the backward equation:

$$\frac{\partial w}{\partial t} = -rw + \alpha P\frac{\partial w}{\partial P} + \beta(\bar{\sigma}-\sigma_0)\frac{\partial w}{\partial \sigma}$$

$$+ \frac{1}{2}\sigma^2 P^2\frac{\partial^2 w}{\partial P^2} + \delta\gamma\sigma P\frac{\partial^2 w}{\partial P\partial\sigma} + \frac{1}{2}\gamma^2\frac{\partial^2 w}{\partial\sigma^2}.$$

The boundary condition for the backward equation is $w(P,\sigma,0) = g(P)$. Now let $g(P_{(t)}) = \max\{0, P_t - c\}$ and set $\alpha = r$ and replace $\beta(\bar{\sigma}-\sigma)$ with $[\beta(\bar{\sigma}-\sigma) - \lambda^*]$. The result is the P.D.E. in (4) that we are trying to solve. Hence risk-neutral valuation with the appropriate adjustments to the σ process solves the P.D.E.