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Delta hedging with the smile

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DELTA HEDGING WITH THE SMILE

Sami Vähämaa (sami@uwasa.fi)

University of Vaasa, Department of Accounting and Finance,

P.O. Box 700, FIN-65101 Vaasa, Finland

Valuable comments by Stewart Hodges, Jussi Nikkinen,

Nikos Nomikos, Bernd Pape, Seppo Pynnönen, an anonymous

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1. Introduction

Practitioners often price and hedge options using the BLACK-SCHOLES (1973) model, although many studies have shown that it does not adequately describe stock price dynamics. In particular, the constant volatility assumption of the BLACK-SCHOLES (BS) model is obviously violated in financial markets.[1] It has been recognized for a long time that volatility is time-varying, and in addition, tends to be inversely related to stock prices (see e.g., BLACK, 1976a). One manifestation of the BS misspecifications is the volatility smile, the variation of implied volatilities across strike prices. Especially the implied volatility of index options seems to decrease with the strike price (see e.g., RUBINSTEIN, 1994 and MAYHEW, 1995), instead of being constant as assumed by the BS model.[2]

This paper is motivated by the inverse relationship between volatility and stock price movements and its implications for delta hedging. Since volatility is an important determinant of hedge ratios, incorrect volatility assumptions may lead to incorrect deltas. If volatility is time-varying and correlated with the price changes of the underlying asset, the delta must control not only for the direct impact of the underlying price change on the option price, but also for the indirect impact of the simultaneous change in volatility. This has been recently noted e.g. in DERMAN et al. (1996), ENGLE and ROSENBERG (2000), COLEMAN et al. (2001), and MIXON (2002). Intuitively, an inverse relation between volatility changes and stock returns would suggest that the BS delta is too large.

The purpose of this paper is to examine whether the delta hedging performance of the BLACK-SCHOLES model can be improved by taking into account the inverse movements between volatility and the underlying stock price. Following DERMAN et al. (1996) and COLEMAN et al. (2001), the inverse relationship between volatility changes and stock returns is inferred from the implied volatility smile. In particular, the slope of the volatility smile is utilized to adjust the Black-Scholes delta. In this study, the resulting adjusted BS delta is referred to as the smile-adjusted delta.

Despite the intensive empirical research on different option pricing models, surprisingly little is known about the hedging performance beyond the BS model. Previous studies on hedging performance have focused on different time-varying volatility option pricing models. The hedging performance of stochastic volatility models is investigated, e.g., in BAKSHI et al. (1997, 2000), NANDI (1998), LIM and GUO (2000), and KIM and KIM (2004) whereas DUMAS et al. (1998), ENGLE and ROSENBERG (2000), COLEMAN et al. (2001), LIM and ZHI (2002), and YUNG and ZHANG (2003) examine option hedging under deterministic volatility models. A bit surprisingly, these studies indicate that although time-varying volatility option pricing models clearly outperform the BS model in terms of pricing (see e.g., BAKSHI et al., 1997 and CORRADO and SU, 1998), such models do not necessarily provide better hedging performance.[3] For instance, BAKSHI et al. (1997) show that stochastic volatility models improve the delta hedging performance of the BS model only when out-of-the-money options are hedged, while DUMAS et al. (1998) and YUNG and ZHANG (2003) document the BS model to outperform deterministic volatility models in terms of delta hedging. Whereas the existing literature has focused on the hedging performance of different time-varying volatility option pricing models, the current study takes a different approach, and contributes to the literature by examining whether the hedging performance of the BS model can be improved with a rather simple adjustment of the BS delta.

The remainder of this paper is organized as follows. The smile-adjusted delta is introduced in Section 2. In Section 3, the FTSE 100 index option data used in the empirical analysis are described. Section 4 presents the methodology applied in the paper. Empirical findings are reported in Section 5. Finally, concluding remarks are offered in Section 6.

2. Smile-Adjusted Delta

The BLACK-SCHOLES (1973) model gives the price and the hedge ratios of an option as functions of the underlying asset price, strike price of the option, risk-free interest rate, time to maturity of the option, and the volatility of the underlying asset. By definition, the volatility parameter is assumed to be a known constant. For a European call option on a non-dividend-paying stock, the BS delta is given by

$$\delta_{BS} = \frac{\partial c(S, K, \sigma, r, T)}{\partial S} = N\left(\frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \quad (1)$$

where $c(\cdot)$ denotes the BS call option pricing formula, $N(\cdot)$ denotes cumulative standard normal distribution function, S is the price of the underlying asset, K is the strike price of the option, σ is the volatility of the underlying asset, r is the risk-free interest rate, and T denotes time to maturity.

An unfortunate inconsistency between the BS model and empirical regularities is the behavior of volatility. The constant volatility assumption of the BS model is indisputably violated in practice. Volatility appears to be time-varying, and in addition, tends to be negatively correlated with changes in the underlying stock price. Therefore, the delta must control not only for the direct impact of the underlying price change on the option price, but also for the indirect impact of the volatility change which is correlated with the underlying price change. Assume volatility to be a deterministic function of S , K , and T . Then, by the chain rule, the delta of the option is given by

$$\delta = \frac{\partial c}{\partial S} + \frac{\partial c}{\partial \sigma} \frac{\partial \sigma}{\partial S} \quad (2)$$

where $\partial c/\partial \sigma$ is the vega of the option. Since vega is always positive, Equation (2) shows that in the case of a negative correlation between stock returns and volatility changes, the delta should be smaller than the BS delta. Unfortu-

nately, the term $\partial\sigma/\partial S$ in Equation (2) is difficult to quantify.

Following DERMAN et al. (1996) and COLEMAN et al. (2001), the unknown dependence of volatility on the underlying stock price, $\partial\sigma/\partial S$, can be approximated by the slope of the volatility smile, $\partial\sigma/\partial K$.^[4] Substituting $\partial\sigma/\partial K$ into Equation (2), the delta of the option can be expressed as

$$\delta_{SAD} = \frac{\partial c}{\partial S} + \frac{\partial c}{\partial \sigma} \frac{\partial \sigma}{\partial K} = \delta_{BS} + v_{BS} \frac{\partial \sigma}{\partial K}. \quad (3)$$

In this study, the delta given by Equation (3) is referred to as the smile-adjusted delta. The smile-adjusted delta utilizes the volatility smile together with the vega of the option to account for the relationship between volatility changes and underlying stock returns. When the smile is downward sloping, the BS delta is adjusted downwards to account for the offsetting movement between volatility and the underlying asset price. Note that in the special case of a completely flat smile, the smile-adjusted delta becomes equal to the BS delta. By approximating $\partial\sigma/\partial S$ with $\partial\sigma/\partial K$, it is assumed that as S changes by one unit, there is a parallel shift of $\partial\sigma/\partial K$ units in the volatility smile. If the current smile is downward sloping, the approximation assumes that the volatility smile is shifted downwards as the price of the underlying stock increases, and thus, all fixed-strike volatilities decrease by the amount defined by the slope of the current smile and at-the-money volatility decreases twice as much.

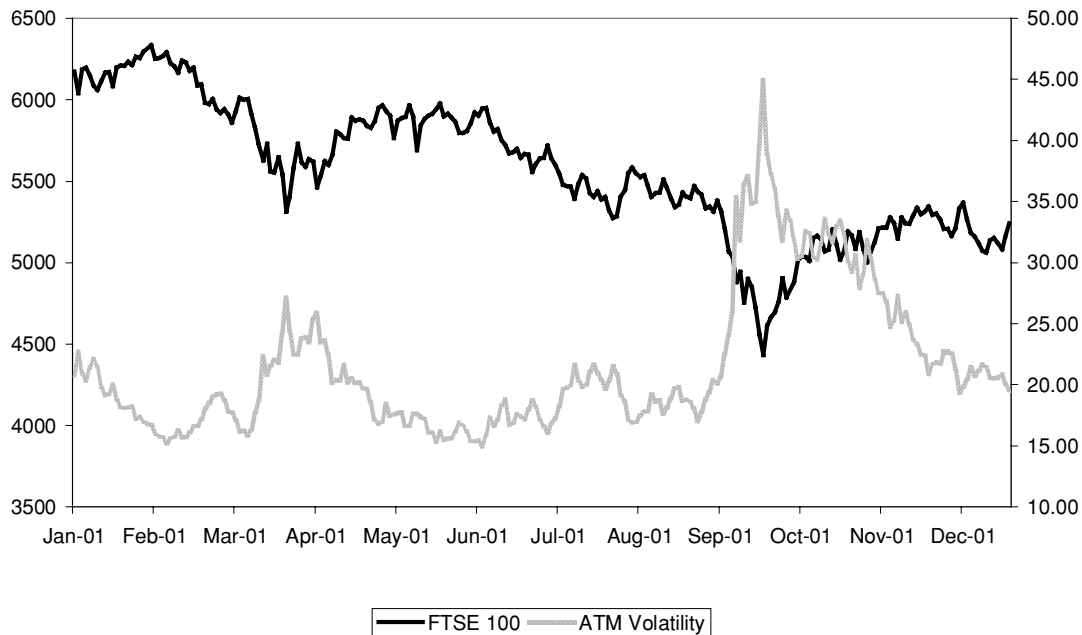
3. Data

The data used in this study contain settlement prices of the FTSE 100 index options traded on the London International Financial Futures and Options Exchange (LIFFE). The FTSE 100 index is a capitalization-weighted index consisting of

the 100 largest companies traded on the London Stock Exchange. Both European and American options on the FTSE 100 index are traded on the LIFFE. In this study, the European-style options (ticker symbol ESX) are used. The market for the European-style FTSE 100 index options is the most active equity options market in the United Kingdom. The European-style FTSE 100 index options have a wide range of strike prices available for trading. The delivery months are March, June, September, and December. In addition, the four nearest calendar months are always available for trading. The FTSE 100 options expire on the third Friday of the delivery month. Dividend adjustments are incorporated via implied index futures, and thus, the FTSE 100 index options are priced as options on futures. The sample period used in this study extends from January 2, 2001 to December 28, 2001. The settlement prices for the FTSE 100 index options and the closing prices for the implied index futures are obtained from the LIFFE.

The risk-free interest rate needed for the calculation of the deltas and for the delta hedging experiment is proxied by the three-month LIBOR (London Interbank Offered Rate) rate. The LIBOR data are obtained from the British Bankers' Association.

The development of the FTSE 100 index and the 1-month at-the-money (ATM) implied volatility during 2001 are presented in Figure 1.^[5] It can be noted that there exists a strong inverse relationship between index level and volatility. Descriptive statistics for the FTSE 100 index and 1-month ATM implied volatility series are presented in Table 1. During the sample period, the index level ranged from 4434 to 6335 points. The mean continuously compounded daily return on the FTSE 100 index was very close to zero and the standard deviation of the daily returns was 0.0136. The standard deviation corresponds to annualized volatility of about 21.60%, which is very close to the mean level of ATM implied volatility of 21.50%. The inverse relationship between index

Figure 1: FTSE 100 Index and At-the-Money Implied Volatility

level and volatility observable in Figure 1 is confirmed by the considerable negative correlation between index returns and volatility changes reported in Panel C of Table 1.

Two exclusionary criteria are applied to the complete FTSE 100 index option sample to construct the final sample used in the empirical analysis. First, options with fewer than 5 or more than 120 trading days to maturity are eliminated from the sample. This choice avoids any expiration-related unusual price fluctuations and minimizes the liquidity problems often affecting the prices of long-term options.^[6] Second, options with moneyness greater 1.10 or less than 0.90 are eliminated. Moneyness is defined as the ratio of futures price to strike price for call options and strike price to futures price for put options. The moneyness criterion is applied because deep out-

of-the-money and in-the-money options tend to be thinly traded.

The final sample contains 35,180 settlement prices on options with 5 to 120 trading days to maturity and moneyness between 0.90 and 1.10. This sample is considered to be a representative sample of the most actively traded option contracts. The sample is partitioned into three moneyness and two time to maturity categories. A call option is said to be out-of-the-money (OTM) if the moneyness ratio is less than 0.97, at-the-money (ATM) if the ratio is larger than 0.97 and less than 1.03, and in-the-money (ITM) if the ratio is greater than 1.03. An option is said to be short-term if it has less than 40 trading days to expiration and long-term otherwise. This maturity and moneyness partitioning produces 12 categories for which the empirical results will be reported.

Table 1: Descriptive Statistics**Panel A: Level**

	FTSE 100 Index	ATM Implied Volatility
Mean	5564.16	21.50
Median	5550.60	19.83
Minimum	4433.70	14.95
Maximum	6334.50	44.93
Standard Deviation	414.04	5.72
Skewness	-0.14	1.45
Excess Kurtosis	-0.67	1.60

Panel B: Logarithmic first differences

	FTSE 100 Index	ATM Implied Volatility
Mean	-0.0007	-0.0002
Median	-0.0005	-0.0006
Minimum	-0.0416	-0.1443
Maximum	0.0398	0.3087
Standard Deviation	0.0136	0.0536
Skewness	-0.0507	0.9058
Excess Kurtosis	0.3620	4.1659

Panel C: Correlation

	FTSE 100 Index	ATM Implied Volatility
FTSE 100 Index	1.00	-0.68
ATM Implied Volatility	-0.68	1.00

Summary statistics for the final sample are reported in Table 2. The reported numbers are (i) the average settlement price, (ii) the average implied volatility, and (iii) the total number of observations in each maturity and moneyness category. The final sample contains almost equal amounts of call and put options and short and long maturity options are also almost equally represented. For moneyness classifications, OTM and ATM options both account for 35% of the total sample and ITM options for the remaining 30%. As expected, Table 2 shows that the average option price increases with moneyness and time to maturity.

4. Methodology

Since the FTSE 100 index options are priced as options on futures, BLACK's (1976b) model for European futures options is applied in the empirical analysis to calculate the deltas, vegas, and implied volatilities. The deltas and vegas for each option are derived from BLACK's model using the exact implied volatility for that particular option as the volatility parameter. Since the volatility smile of index options tends to be approximately linear (see e.g. RUBINSTEIN, 1994; DERMAN, 1999), $\partial\sigma/\partial K$ is estimated by fitting a linear model of volatility as a function of the strike price based on least squares criterion.[7]

Table 2: FTSE 100 Index Option Sample

Moneyness	Time to Maturity		Subtotal
	Short	Long	
OTM	28.21	86.91	12233
	21.94	21.20	
	5980	6253	
ATM	136.93	227.58	12273
	21.17	21.21	
	5981	6292	
ITM	394.92	455.67	10674
	21.84	21.21	
	5225	5449	
Subtotal	17186	17994	35180

The quantitative fit of the BS and smile-adjusted deltas is tested based on a first-order TAYLOR series expansion with the following two regression specifications, respectively

$$\Delta c = \beta_0 + \beta_1 \left[\frac{\partial c}{\partial S} \Delta S \right] + \varepsilon \tag{4}$$

$$\Delta c = \beta_0 + \beta_1 \left[\frac{\partial c}{\partial S} \Delta S \right] + \beta_2 \left[\frac{\partial c}{\partial \sigma} \frac{\partial \sigma}{\partial K} \Delta S \right] + \varepsilon \tag{5}$$

where c denotes the option price, S is the price of the underlying asset, K is the strike price of the option, and Δ denotes the first difference operator. An accurate delta estimate should produce a very small negative β_0 , and β_1 of one, with the regression R^2 equal to one. The accuracy of the BS and smile-adjusted deltas is compared based on the R^2 of the regressions; the more accurate delta should produce a higher R^2 . Furthermore, since Equation (4) is a restricted version of Equation (5), the Wald chi-square statistic is applied to test whether the contribution of the adjustment term is statistically significant.

In order to examine the delta hedging performance of the BS and smile-adjusted deltas, a self-financed delta-hedged portfolio with one unit short position in an option, δ units of the underlying asset, and B units of a risk-free bond is

constructed. The value of the portfolio Π at time t is

$$\Pi_t = \delta_t S_t + B_t - c_t. \tag{6}$$

At the beginning of the hedging horizon $B_0 = c_0 - \delta_0 S_0$, and thus, $\Pi_0 = 0$. The delta hedging performance of the two deltas is examined in 1-day, 5-day, and 10-day hedging horizons using daily rebalancing of the hedge portfolio. At each hedge-revision time t the hedge parameter is recomputed and the position in the bond is adjusted to

$$B_t = e^r B_{t-1} + S_t (\delta_{t-1} - \delta_t). \tag{7}$$

The delta hedging error ε from hedge-revision time $t-1$ to t is calculated as

$$\varepsilon_t = \delta_{t-1} S_t - c_t + e^r B_{t-1}. \tag{8}$$

and the total hedging error during the hedging horizon τ is given by Π_τ , the value of the portfolio at the end of the hedging horizon

$$\varepsilon_\tau = \sum_{t=1}^{\tau} \varepsilon_t = \Pi_\tau. \tag{9}$$

The delta hedging performance of the BS and smile-adjusted deltas is analyzed based on two commonly used error statistics, (i) mean absolute

hedging error (MAHE) and (ii) root mean squared hedging error (RMSHE). The error statistics are calculated as

$$\text{MAHE} = \frac{1}{n} \sum_{i=1}^n |\varepsilon_{\tau_i}| \quad (10)$$

$$\text{RMSHE} = \sqrt{\frac{1}{n} \sum_{i=1}^n \varepsilon_{\tau_i}^2}. \quad (11)$$

To avoid any distributional assumptions about the error statistics, bootstrapping is applied to test whether the hedging errors under the BS and smile-adjusted deltas are statistically significantly different. Since the hedger's main objective is risk minimization, RMSHE is considered to be the primary evaluation criterion.

5. Empirical Results

A comparison of the average BLACK-SCHOLES and smile-adjusted deltas for different moneyness categories is presented in Table 3. Since the volatility smile of index options is usually downward sloping and the vega for both call and put options is always positive, the smile-adjusted deltas are consistently smaller than the corresponding BS deltas. On average, the difference between the two deltas is about 0.05. Since the vega is largest for ATM options, the difference between the two deltas is naturally greatest among ATM options.

Surprisingly little attention in the existing literature has been devoted to the size of the delta. DERMAN (1999) suggests that the size of the delta should be conditional on market conditions. In particular, in stable market conditions when volatility appears to be independent of the underlying index level, the delta should be equal to the BS delta. In trending markets, the volatility may be positively related to the index, and hence the correct delta should be larger than the BS delta. Finally, DERMAN (1999) suggests that in highly volatile market conditions when volatility and the index are likely to move in the opposite directions, a smaller than BS delta should be optimal for hedging purposes. MIXON (2002) argues that the delta should be smaller than the BS delta in order to be consistent with the empirical regularities of volatility dynamics. Previously, deltas that are different from the BS delta have been empirically documented in BAKSHI et al. (2000), COLEMAN et al. (2001), and LIM et al. (2002). BAKSHI et al. (2000) show that the deltas produced by stochastic volatility models are smaller than the BS delta for OTM put options, but larger for ITM put options. COLEMAN et al. (2001) document the deltas based on a deterministic volatility model to be consistently smaller than the BS deltas. LIM et al. (2002) report that the deltas based on the DERMAN-KANI (1994) implied tree model are smaller than the BS deltas, while the deltas derived from JACK-WERTH's (1997) model tend to be larger than the BS deltas.

Table 3: Comparison of the Black-Scholes and Smile-Adjusted Deltas

Moneyness	Call Options			Put Options		
	BS	SAD	Difference	BS	SAD	Difference
Full Sample	0.46	0.41	0.05	-0.50	-0.54	0.04
OTM	0.17	0.14	0.04	-0.20	-0.25	0.05
ATM	0.50	0.44	0.06	-0.48	-0.54	0.06
ITM	0.77	0.73	0.04	-0.80	-0.84	0.04

To compare the stability of the Black-Scholes and smile-adjusted deltas, the hedge ratios are plotted in Figure 2. The deltas are presented for the second shortest maturity call option and for the strike price that is closest to the underlying index value on the first trading day of the month and is held constant until the first trading day of the following month. As can be seen from Figure 2, the BS and smile-adjusted deltas tend to move in parallel. Apart from the size difference of the deltas also noted in Table 3, the two deltas seem rather similar. In particular, no differences in the stability of the hedge ratios may be observed from Figure 2. Further analysis of the daily changes in the deltas confirms that the BS and smile-adjusted deltas behave very similarly over time. The variance of the daily changes in the Black-Scholes delta is 0.0023, while the corresponding variance for the smile-adjusted delta is 0.0025. A simple F -test for

equality of the variances suggests that this difference in variances is statistically insignificant.

Table 4 reports the regression results based on Equations (4) and (5) for different moneyness and maturity categories. BS and SAD in column headings denote the BLACK-SCHOLES and smile-adjusted deltas, respectively. The reported R^2 for each regression is the adjusted R^2 . White's heteroskedasticity consistent standard errors are presented in parentheses below the coefficient estimates. The null hypothesis in the reported Wald chi-square test is $\beta_2 = 0$.

Several interesting features can be noted from Table 4. Most importantly, the R^2 values under the BS delta are consistently lower than under the smile-adjusted delta, indicating a better quantitative fit of the latter. For the full sample, the BS model can explain 96.51% of the observed option price changes whereas the smile-adjusted delta

Figure 2: Comparison of the Black-Scholes and Smile-Adjusted Deltas

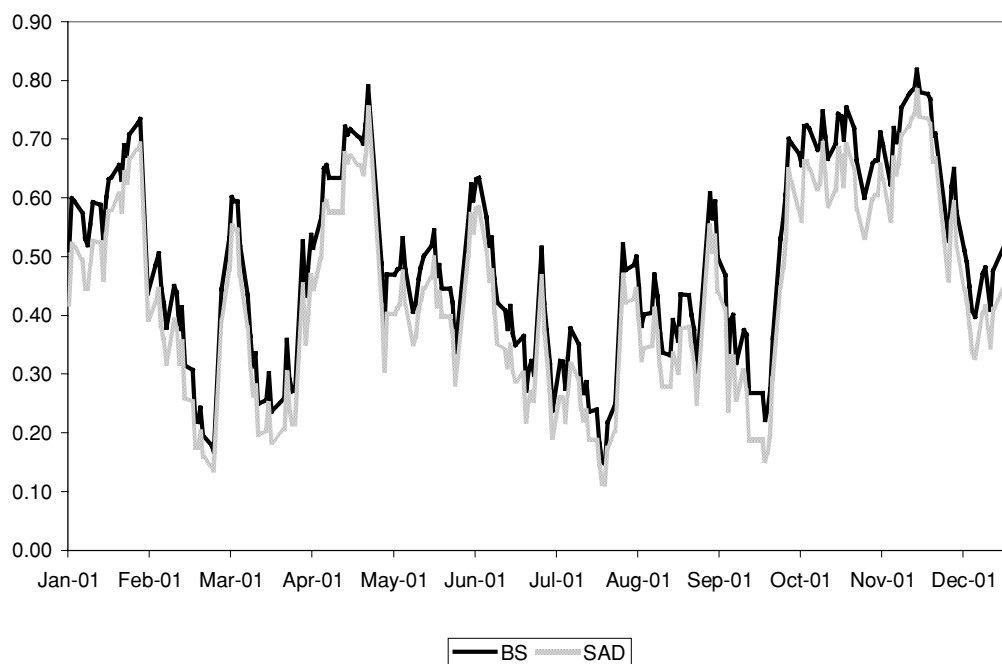


Table 4: Quantitative Fit of the Black-Scholes and Smile-Adjusted Deltas

Moneyness	Time to Maturity	BS			SAD				Wald Test p-value
		β_0	β_1	R^2	β_0	β_1	β_2	R^2	
Full Sample	All	-0.49 (0.04)	0.99 (0.00)	0.9651	-0.56 (0.03)	1.00 (0.00)	0.76 (0.02)	0.9717	0.00
	Short	-0.38 (0.06)	0.99 (0.00)	0.9621	-0.39 (0.05)	0.99 (0.00)	0.92 (0.05)	0.9684	0.00
	Long	-0.60 (0.05)	0.99 (0.00)	0.9684	-0.71 (0.04)	1.00 (0.00)	0.68 (0.02)	0.9756	0.00
OTM	All	-0.87 (0.05)	0.87 (0.01)	0.8871	-0.87 (0.05)	0.88 (0.01)	0.72 (0.03)	0.9107	0.00
	Short	-0.85 (0.08)	0.80 (0.02)	0.8422	-0.82 (0.07)	0.82 (0.02)	0.91 (0.06)	0.8780	0.00
	Long	-0.95 (0.07)	0.90 (0.01)	0.9148	-0.99 (0.07)	0.92 (0.01)	0.64 (0.03)	0.9339	0.00
ATM	All	-0.80 (0.07)	0.97 (0.01)	0.9527	-0.85 (0.07)	0.98 (0.00)	0.74 (0.03)	0.9608	0.00
	Short	-0.91 (0.12)	0.97 (0.01)	0.9431	-0.90 (0.11)	0.97 (0.01)	0.85 (0.07)	0.9511	0.00
	Long	-0.71 (0.09)	0.98 (0.01)	0.9624	-0.79 (0.08)	0.98 (0.01)	0.67 (0.03)	0.9708	0.00
ITM	All	-0.49 (0.06)	1.03 (0.00)	0.9858	-0.55 (0.05)	1.03 (0.00)	0.75 (0.04)	0.9887	0.00
	Short	-0.30 (0.09)	1.03 (0.00)	0.9865	-0.30 (0.08)	1.03 (0.00)	0.94 (0.09)	0.9892	0.00
	Long	-0.67 (0.08)	1.03 (0.00)	0.9850	-0.78 (0.07)	1.02 (0.00)	0.66 (0.04)	0.9883	0.00

explains 97.17% of the changes. In general, the quantitative fit of both deltas is relatively high for all moneyness and maturity categories, with R^2 's ranging from 84.22% to 98.65% under the BS delta and from 87.80% to 98.92% under the smile-adjusted delta.

The regression results reported in Table 4 also demonstrate that the difference between the quantitative fit of the two models is largest among short-maturity OTM options and smallest for short-maturity ITM options. Both models seem to fit ITM options better than ATM and OTM options. The β_1 estimates of both models are quite close to unity for ATM and ITM options. However, for OTM options the β_1 estimates are considerably below unity. In most cases, there are

no significant differences in the β_1 estimates under the two models, but especially for OTM options, the β_1 estimates under the smile-adjusted delta are slightly closer to unity. The coefficient for the smile-adjustment term in Equation (5) appears highly statistically significant in all regressions. Furthermore, Wald's chi-square test indicates that the regression models for the BS and smile-adjusted deltas are statistically significantly different from each other, thereby suggesting that the smile-adjusted delta is a significant improvement over the standard Black-Scholes delta.

The results of the delta hedging experiment are reported in Tables 5, 6, and 7. Again, BS and SAD in column headings denote the BLACK-

Table 5: Hedging Errors for the 1-Day Hedging Horizon

Moneyness	Time to Maturity	MAHE				RMSHE			
		BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	4.05	3.99	0.06	(1.52)**	6.92	6.31	0.61	(9.59)**
	Short	4.03	3.79	0.24	(6.48)**	7.43	6.79	0.64	(9.48)**
	Long	4.06	4.18	-0.12	(-2.81)**	6.38	5.81	0.57	(9.74)**
OTM	All	3.47	3.38	0.09	(2.56)**	6.15	5.51	0.64	(11.63)**
	Short	3.19	2.97	0.22	(7.34)**	6.37	5.67	0.70	(12.36)**
	Long	3.74	3.78	-0.04	(-1.07)	5.93	5.35	0.58	(10.85)**
ATM	All	4.93	4.86	0.07	(1.57)*	8.03	7.39	0.64	(8.57)**
	Short	5.31	4.98	0.33	(6.56)**	8.91	8.27	0.64	(7.75)**
	Long	4.57	4.73	-0.16	(-3.48)**	7.08	6.44	0.64	(9.84)**
ITM	All	3.61	3.60	0.01	(0.31)	6.22	5.68	0.54	(9.50)**
	Short	3.41	3.24	0.17	(5.41)**	6.49	5.87	0.62	(10.56)**
	Long	3.80	3.95	-0.15	(-3.73)*	5.95	5.49	0.46	(8.31)**

** significant at the 0.01 level

* significant at the 0.05 level

SCHOLES and smile-adjusted deltas, respectively. The reported error statistics for each maturity-moneyness category are the mean absolute hedging error (MAHE) and the root mean squared hedging error (RMSHE). The third and seventh columns of the tables report the mean differences between the delta hedging errors of the two models, while the mean percentage differences are shown (in parentheses) in the fourth and eighth columns. Bootstrapping is applied to test whether the differences in the hedging errors are statistically significant.

The hedging results for the 1-day hedging horizon are given in Table 5. A general feature observable in Table 5 (and also in Tables 6 and 7) is that the hedging errors are larger for ATM options than for OTM and ITM options. This is reasonable, considering that the gamma and vega of an option are at maximum for ATM options. Similarly, since the gamma of the option decreases as the maturity of the option increases, it may be expected a priori that the delta hedging errors would be smaller for longer maturity options. This pattern is indeed observable in the reported root mean squared hedging errors.

Turning the focus onto the hedging performance comparison, the results reported in Table 5 clearly indicate that the smile-adjusted delta is a substantial improvement over the BS delta. For the full sample, the smile-adjusted delta outperforms the BS delta in terms of hedging performance based both on MAHE and RMSHE. The difference in both error statistics is statistically significant at the 1% level. The outperformance of the smile-adjusted delta is most distinct for short-term OTM and ATM options. For all moneyness and maturity categories, the RMSHE under the BS delta is consistently about 10 % larger than under the smile-adjusted delta but the MAHE criterion indicates that the BS delta is more effective for hedging long-term options. However, since the hedger's main objective is risk minimization, the emphasis in the interpretation of the results should be given to the RMSHE.

Further analysis of the hedging errors reported in Table 5 suggests that regardless of the moneyness and maturity of the options, the mean hedging error is slightly smaller under the BS delta while the standard deviation of hedging errors is smaller under the smile-adjusted delta. The analysis also

indicates that the distribution of the hedging errors is highly nonnormal. Both deltas produce negatively skewed and leptokurtic hedging error distributions. Under the BS delta, the distribution of hedging errors is more negatively skewed and also considerably more leptokurtic than under the smile-adjusted

delta, thereby indicating a greater likelihood of extreme hedging errors under the BS delta. This is also shown in the considerably larger root mean squared hedging errors produced by the BS delta.

Table 6 presents the hedging results for the 5-day hedging horizon. The error statistics reported in

Table 6. Hedging Errors for the 5-Day Hedging Horizon

Moneyiness	Time to Maturity	MAHE				RMSHE			
		BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	12.42	10.05	2.37	(23.55)**	18.18	15.66	2.52	(16.14)**
	Short	11.24	9.28	1.96	(21.13)**	18.36	16.52	1.84	(11.12)**
	Long	13.54	10.79	2.75	(25.52)**	18.01	14.79	3.22	(21.81)**
OTM	All	9.95	8.46	1.49	(17.59)**	14.02	12.28	1.74	(14.12)**
	Short	8.01	6.78	1.23	(18.13)**	12.81	11.50	1.31	(11.40)**
	Long	11.78	10.05	1.73	(17.24)**	15.07	12.98	2.09	(16.10)**
ATM	All	17.07	12.99	4.08	(31.44)**	24.63	21.00	3.63	(17.27)**
	Short	17.18	13.84	3.34	(24.12)**	26.53	24.17	2.36	(9.78)**
	Long	16.97	12.19	4.78	(39.17)**	22.72	17.55	5.17	(29.44)**
ITM	All	10.26	8.73	1.53	(17.57)**	13.70	11.94	1.76	(14.76)**
	Short	8.64	7.30	1.34	(18.32)**	12.12	10.25	1.87	(18.25)**
	Long	11.83	10.11	1.72	(17.05)**	15.08	13.38	1.70	(12.72)**

** significant at the 0.01 level

* significant at the 0.05 level

Table 7: Hedging Errors for the 10-Day Hedging Horizon

Moneyiness	Time to Maturity	MAHE				RMSHE			
		BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	20.34	19.42	0.92	(4.71)*	32.63	26.90	5.73	(21.29)**
	Short	19.91	18.34	1.57	(8.54)**	33.22	27.24	5.98	(21.93)**
	Long	20.65	20.20	0.45	(2.21)	32.20	26.65	5.55	(20.81)**
OTM	All	18.92	17.52	1.40	(7.99)**	32.43	25.65	6.78	(26.43)**
	Short	17.82	16.29	1.53	(9.41)*	32.75	26.07	6.68	(25.66)**
	Long	19.73	18.43	1.30	(7.05)	32.18	25.33	6.85	(27.04)**
ATM	All	22.85	21.73	1.12	(5.18)	34.77	29.16	5.61	(19.24)**
	Short	23.13	20.98	2.15	(10.24)*	35.41	29.32	6.09	(20.77)**
	Long	22.66	22.24	0.42	(1.89)	34.32	29.04	5.28	(18.16)**
ITM	All	18.44	18.39	0.05	(0.24)	29.58	24.93	4.65	(18.67)**
	Short	17.97	17.16	0.81	(4.70)	30.54	25.61	4.93	(19.28)**
	Long	18.78	19.29	-0.51	(-2.64)	28.86	24.42	4.44	(18.17)**

** significant at the 0.01 level

* significant at the 0.05 level

Table 6 clearly indicate that hedging under the smile-adjusted delta is more effective than under the BS delta. Regardless of the moneyness and maturity of the options, both error statistics are lower for the smile-adjusted delta than for the BS delta. All differences reported in Table 6 are statistically significant at the 1% level. In contrast to the results in Table 5, the outperformance of the smile-adjusted delta seems to be more apparent in the case of long-term options.

The hedging results for the 10-day hedging horizon are presented in Table 7. In general, the hedging errors for the 10-day horizon are very similar to the errors for the 1-day and 5-day horizons in Tables 5 and 6. Again, the smile-adjusted delta consistently outperforms the BS delta in terms of hedging performance, regardless of the moneyness and maturity of the options. For instance, the RMSHE seems to be around 20 % larger under the BS delta than under the smile-adjusted delta. If compared to the hedging errors for the 1-day hedging horizon reported in Table 5, the results in Tables 6 and 7 indicate that the outperformance of the smile-adjusted delta is more prominent for longer hedging horizons.

To further investigate the hedging performance of the smile-adjusted delta relative to the BS delta, the delta hedging experiment for the 1-day hedging horizon is repeated in two distinct market conditions, in the stable market of May 2001 and in the highly volatile market of September 2001. This analysis is motivated by the “regimes of volatility” model of DERMAN (1999). In May 2001, the FTSE 100 index as well as the implied volatility of the index were relatively stable and the correlation between the index returns and volatility changes was low, -0.16 , and statistically insignificant. In contrast, September 2001 was characterized by extensive movements both in the index and in the implied volatility. In September 2001, the correlation between the index returns and volatility changes was -0.76 and statistically significant at the 1% level. According to DERMAN (1999), the optimal delta should be equal to the BS delta during stable market conditions, whereas in highly volatile markets the optimal delta should be smaller than the BS delta.

Table 8 presents the hedging results for the stable market conditions. The results seem very similar to the results reported for the total sample in

Table 8: Hedging Errors in Stable Market Conditions

Moneyness	Time to Maturity	MAHE				RMSHE			
		BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	2.79	2.75	0.04	(1.63)	4.07	3.92	0.15	(3.72)
	Short	2.64	2.49	0.15	(6.00) **	3.96	3.69	0.27	(7.14) **
	Long	2.96	3.04	-0.08	(-2.44)	4.19	4.17	0.02	(0.57)
OTM	All	2.17	2.13	0.04	(1.65)	3.31	3.17	0.14	(4.40)
	Short	1.79	1.70	0.09	(5.11)	2.90	2.70	0.20	(7.30)
	Long	2.59	2.62	-0.03	(-0.91)	3.72	3.62	0.10	(2.53)
ATM	All	3.75	3.66	0.09	(2.42)	5.07	4.88	0.19	(4.01)
	Short	3.91	3.65	0.26	(7.31) **	5.25	4.88	0.37	(7.67) **
	Long	3.56	3.67	-0.11	(-3.09)	4.86	4.88	-0.02	(-0.33)
ITM	All	2.36	2.35	0.01	(0.11)	3.44	3.36	0.08	(2.26)
	Short	2.08	2.00	0.08	(3.95)	3.07	2.92	0.15	(5.16)
	Long	2.67	2.75	-0.08	(-3.02)	3.81	3.79	0.02	(0.28)

** significant at the 0.01 level

* significant at the 0.05 level

Table 9: Hedging Errors in Highly Volatile Market Conditions

Moneyiness	Time to Maturity	MAHE				RMSHE			
		BS	SAD	Difference	(%)	BS	SAD	Difference	(%)
Full Sample	All	12.76	10.66	2.10	(19.69)**	19.48	16.70	2.78	(16.64)**
	Short	13.39	11.59	1.80	(15.50)**	21.50	19.02	2.48	(13.09)**
	Long	12.02	-9.56	2.46	(25.66)**	16.79	13.48	3.31	(24.56)**
OTM	All	11.55	-9.48	2.07	(21.75)**	17.76	14.77	2.99	(20.20)**
	Short	11.73	-9.90	1.83	(18.44)**	19.19	16.27	2.92	(17.92)**
	Long	11.33	-8.98	2.35	(26.08)**	15.90	12.77	3.13	(24.48)**
ATM	All	14.58	12.34	2.24	(18.12)**	21.78	19.14	2.64	(13.81)**
	Short	15.98	14.10	1.88	(13.32)**	24.60	22.47	2.13	(9.44)**
	Long	12.96	10.32	2.64	(25.65)**	18.01	14.37	3.64	(25.27)**
ITM	All	11.62	-9.68	1.94	(20.13)**	17.92	15.07	2.85	(18.90)**
	Short	11.75	10.10	1.65	(16.34)**	19.39	16.63	2.76	(16.58)**
	Long	11.47	-9.17	2.30	(25.17)**	15.97	12.94	3.03	(23.40)**

** significant at the 0.01 level

* significant at the 0.05 level

Table 5. The RMSHE is consistently lower under the smile-adjusted delta whereas the MAHE suggests that the BS delta provides better hedges for long-term options. However, in most cases the reported differences are statistically insignificant. The hedging results for the highly volatile market conditions are reported in Table 9. For highly volatility markets, both error statistics indicate considerable outperformance of the smile-adjusted delta. Regardless of the moneyness and maturity of the options, the smile-adjusted delta always leads to smaller hedging errors in highly volatile market conditions. All differences shown in Table 9 are statistically significant at the 1% level. In general, the results in Tables 8 and 9 are quite consistent with DERMAN's (1999) model. Although the only statistically significant differences in Table 8 indicate outperformance of the smile-adjusted delta, the BS delta still seems quite appropriate for hedging purposes. On the other hand, Table 9 clearly demonstrates that in highly volatile market conditions the correlation between volatility and stock index movements must be taken into account.

6. Conclusions

The constant volatility assumption of the BLACK-SCHOLES (1973) model is indisputably violated in practice. Volatility appears to be time-varying, and moreover, tends to be negatively correlated with stock returns. Still, the BS model is the most common framework for pricing and hedging options, mainly due to its simplicity and tractability. This paper is motivated by the inverse relationship between volatility and stock price movements and its implications for delta hedging. If volatility is time-varying and correlated with the underlying stock returns, the delta must control not only for the direct impact of the underlying price change on the option price, but also for the indirect impact of simultaneous change in volatility. Intuitively, an inverse relation between volatility and stock returns would suggest that the BS delta is too large.

This study examines whether the delta hedging performance of the BLACK-SCHOLES model can be improved by taking into account the inverse movements between volatility and the underlying stock price. In particular, a rather simple

adjustment of the BS delta is investigated. Following DERMAN et al. (1996) and COLEMAN et al. (2001), the volatility smile is utilized to adjust the BS delta to account for the inverse relationship between volatility changes and stock returns.

Empirical tests in the FTSE 100 index option market show that the delta hedging performance of the BS model can be substantially improved by adjusting the BS delta to account for the inverse movements between volatility and stock prices. The smile-adjusted delta consistently leads to smaller hedging errors, and thus outperforms the BS delta in terms of hedging performance. The outperformance of the smile-adjusted delta is most distinct for short-term OTM and ATM options and becomes more prominent as the hedging horizon lengthens.

Since short-term OTM and ATM options tend to be the most actively traded contracts in index option markets, the findings reported in this paper have significant practical relevance. The practical relevance of the results is further enhanced by the fact that the BS deltas, vegas, and smiles are continuously monitored by practitioners, and thus the smile-adjusted delta should be simple to implement for practical purposes. In general, the findings reported in this paper have important implications for risk management of options as they indicate the importance of the correlation between volatility changes and stock returns for appropriate risk management. Furthermore, the empirical results demonstrate that due to the inverse relation between volatility and stock price movements, the correct delta is smaller than the BS delta.

ENDNOTES

- [1] A substantial body of literature has been devoted to the development of option pricing models with time-varying volatility. The BLACK-SCHOLES constant volatility assumption is relaxed in stochastic volatility models such as HULL and WHITE (1987) and HESTON (1993), in the deterministic volatility models of DUPIRE (1994), DERMAN and KANI (1994), and RUBINSTEIN (1994), and in the ARCH models of DUAN (1995) and HESTON and NANDI (2000).
- [2] Since implied volatility of index options is decreasing with the strike price, the volatility curve looks more like a smirk than a smile. Therefore, besides the volatility smile, the curve is also commonly referred to as volatility smirk and volatility skew.
- [3] Time-varying volatility models outperform the BLACK-SCHOLES model in terms of pricing, but are still computationally too demanding for real-time option pricing and even more so for calculating hedge ratios.
- [4] The deterministic volatility models of DUPIRE (1994), DERMAN et al. (1994) and RUBINSTEIN (1994) also infer the dependence of volatility on the underlying asset price from the volatility smile.
- [5] The 1-month ATM implied volatility series is obtained by linear interpolation between the two adjacent maturity ATM implied volatilities. Three shortest maturity option series are used as follows. The shortest and the second shortest maturity options are used until the shortest has 5 days to maturity. Thereafter, the second and the third shortest maturity options are used until the expiry of the shortest maturity option.
- [6] The two shortest maturity contracts are the most actively traded ones.
- [7] The linear parameterisation of the volatility smile is considered to be adequate for stock index options. However, it should be noted that a quadratic model may be more appropriate for the more convex volatility smiles typically observed for individual stock options.

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